Two Cylinders Problem: Solution

There are complex algebraic and calculus-based solutions for this problem, but Archimedes (278 - 212 BC) (is that a long time ago or what!) had worked out a solution. We're talking well before Sir Isaac Newton (1642 - 1726) and the development of calculus.

Here we present a simple geometric approach. Hope you like it!

Figure 1 shows the two intersecting cylinders.

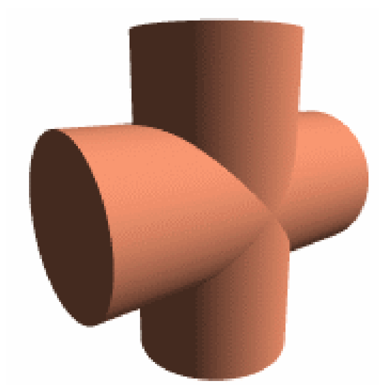

Figure 1. Two Intersecting Cylinders

The volume that is common to both cylinders is the strange looking object shown in Figure 2 called a Steinmetz Solid.

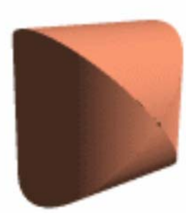

Figure 2. Steinmetz Solid

If you slice the cylinders with a plane parallel to both cylinder axes you will <u>always</u> get a square. Inside this square you can inscribe a circle as shown in Figure 3.

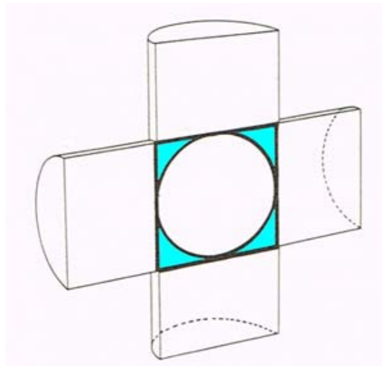

Figure 3. Cross Section

If you start taking slices parallel to the axes, and move the plane out away from the center what happens? In every case the intersection will still be a square, and you can inscribe a circle in it. This is illustrated in Figure 4.

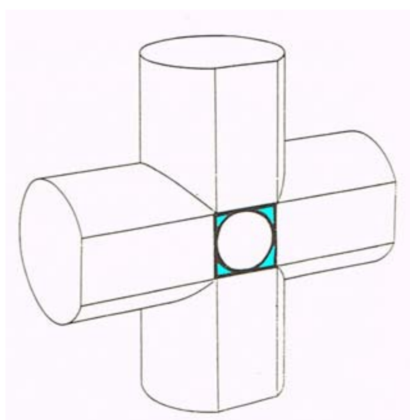

Figure 4. Cross Section Away from the Center

Now, here's the trick. If these slices were thin enough and you stacked them all together you would get the Steinmetz Solid with a sphere inside. How do you know it would be a sphere and not just some kind of elongated circular thing? Just take a slice at the center perpendicular to the plane we have been working in. What do you get? A circle formed by the horizontal cylinder. How about that!

So, all we need to know is the volume of the sphere and how much bigger the volume of the object is.

Figure 5 below shows a square with an inscribed circle.

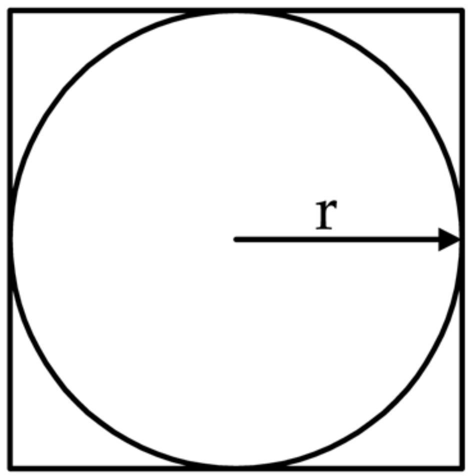

Figure 5. Square with Inscribed Circle

The area of the square is $4r^2$ (because each side of the square is $2r$).
The area of the circle is πr^2

If we let R be the ratio of the area of the square to the area of the circle we get: $R = 4r^2/\pi r^2$ or

$R = 4/\pi$

In other words, $4/\pi$ is how much bigger the area of the square is compared to the inscribed circle. How about the volume of the object compared to the volume of the sphere? Same ratio!

OK, so here is the last bit.

The volume of the sphere is $(4/3)\pi r^3$
If we multiply the volume of the sphere by R we get:

Volume of Steinmetz Solid = $(4/3)\ \pi r^3 \times 4/\pi = 16r^3/3$